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# On superspace Chern-Simons-like terms

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Abstract: We search for superspace Chern-Simons-like higher-derivative terms in the low energy effective actions of supersymmetric theories in four dimensions. Superspace Chern-Simons-like terms are those gauge-invariant terms which cannot be written solely in terms of field strength superfields and covariant derivatives, but in which a gauge potential superfield appears explicitly. We find one class of such four-derivative terms with $N = 2$ supersymmetry which, though locally on the Coulomb branch can be written solely in terms of field strengths, globally cannot be. These terms are classified by certain Dolbeault cohomology classes on the moduli space. We include a discussion of other examples of terms in the effective action involving global obstructions on the Coulomb branch.

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## 1. Introduction

Current fundamental non-gravitational theories of nature are effective field theories — local, Lorentz-invariant, low energy approximations to some complete theory. Effective theories are organized in a derivative expansion, where terms in the action with fewer derivatives dominate the long wavelength, low energy behavior. This expansion is organized by assigning spacetime derivatives weight +1 and fields various other weights (which we will discuss below). One then considers all terms of a given weight that can appear in the action consistent with any gauge invariances as well as global symmetries. The lowest-weight terms are the most important at low energies. This expansion is useful when there are only a finite number of terms of a given weight. If a field should have negative weight, then the derivative expansion breaks down. In this paper we will show that the derivative expansion in four-dimensional theories with extended supersymmetry suffers from this problem: vector potential multiplets have non-positive weight. But the way vector potentials enter into the action is constrained by gauge invariance, so there may be, in fact, only a finite number of gauge-invariant terms of a given weight. The problem that this paper faces is how to list all gauge-invariant terms of a given weight if the gauge potential does not have positive weight.

Let us illustrate this problem in a simple, non-supersymmetric, context. Consider a theory of a single abelian vector field, $A_\mu$, in four dimensions. Normally, $A_\mu$ can be assigned a positive weight, e.g., $w(A_\mu) = +1$, the same as its scaling dimension. In this case there are a finite number of terms of a given weight, even without using gauge invariance. But in theories with extended supersymmetry, we will see that we must assign weight $w(A_\mu) = 0$. Then there are an infinite number of local, Lorentz-invariant terms for a given weight, before imposing gauge invariance. A commonly-held belief is that there are no Chern-Simons-like terms in even dimensions: all gauge invariants made just of abelian gauge fields can be written solely in terms of field strengths, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and their derivatives. If



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this is true, then, since $F_{\mu\nu}$ has weight $+1$, there will be only a finite number of gauge-invariant terms of a given weight. However, we are unaware of a proof of the absence of Chern-Simons-like terms in even dimensions. To prove it one should show that for every gauge-invariant $f$ there exists a $g$ such that $\int d^{2n}x\, f(A_\mu, \partial_\nu) = \int d^{2n}x\, g(F_{\mu\nu}, \partial_\rho)$ modulo surface terms. This is difficult to prove because the number of ways that a $g(F_{\mu\nu}, \partial_\rho)$ can be written using integration by parts as some not obviously gauge-invariant collection of terms $f(A_\mu, \partial_\nu)$ grows at least exponentially with the number of $F$'s in $g$.

This example gives a flavor of the type of problem that we will face in supersymmetric effective actions. We emphasize, though, that in the supersymmetric context, the existence of *superfield* Chern-Simons-like terms does *not* necessarily imply the existence of the hypothetical non-supersymmetric Chern-Simons-like terms discussed above. Indeed, examples of superspace Chern-Simons-like terms which do not are known, as will be discussed below.

The outline of this paper is as follows. In the next section we present a general discussion of Chern-Simons-like terms in effective actions in four dimensions, and show that for theories with $N \geq 2$ supersymmetry their existence is a logically pressing issue for carrying out a systematic derivative expansion.

In sections 3 and 4 we carry out a search for such Chern-Simons-like terms in $N = 2$ theories, following two different algebraic strategies which are outlined in section 2. The results are partial and mainly negative, except for one class of 4-derivative terms found in section 3 which is superspace Chern-Simons-like only globally on the Coulomb branch of $N = 2$ theories.

Finally, in section 5 we conclude with some comments on the new term found in section 3. It corresponds to a Dolbeault cohomology class on the Coulomb branch, some examples of which are given. For the reader who wishes to see only the new term, the negative results of sections 3 and 4 can probably be skipped without much loss of comprehensibility. Section 5 also discusses a class of holomorphic 4-derivative terms which may exist by virtue of other global obstructions on the Coulomb branch.

## 2. Derivative expansions, gauge invariance, and extended supersymmetry

How we assign weights to the fields is of central importance. This assignment must be compatible with any global symmetries. In particular, if space-time derivatives have weight $+1$, then the supersymmetry algebra implies that the supercharges must be assigned weight $+1/2$, fixing in turn the relative weights of fields within the same supermultiplet in supersymmetric theories. Thus, for $N = 1$ supersymmetry, if a scalar is assigned weight $w(\phi)$, then its fermionic partner in the chiral multiplet must have weight $w(\phi) + 1/2$; likewise, if the gauge potential has weight $w(A_\mu)$, then the gaugino will have weight $w(A_\mu) + 1/2$. For $N = 2$ supersymmetry, the hypermultiplet weights are as in the $N = 1$ chiral multiplet, while the scalar and vector fields in the vector multiplet must have the same weight $w(\phi) = w(A_\mu)$, and the fermions weight greater by $1/2$. $N = 4$ supersymmetric theories have the same weight assignment as in the $N = 2$ vector multiplet.

In theories with a moduli space of inequivalent vacua labelled by the expectation values of scalar fields, one must assign weight 0 to the scalars in order to study the effective action



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as a function on the moduli space. This weight assignment, which is not the same as the canonical scaling dimensions of the fields, has been used repeatedly in studies of effective actions with extended supersymmetries [1, 2, 3, 4]. It implies, in particular, that for $N \geq 2$ theories $w(\phi) = w(A_\mu) = 0$, and $w(\psi_\alpha) = 1/2$, leading to the problem of characterizing or disproving the existence of Chern-Simons-like terms, as explained above.

In a superfield formalism, the existence of Chern-Simons-like terms is more subtle. Superfields are needed to carry out general derivative expansions while preserving supersymmetry. For in an on-shell and/or component formalism, systematic expansions become very difficult because one must self-consistently correct the supersymmetry transformation rules order by order in the derivative expansion at the same time that one tries to construct the supersymmetry invariant higher-order terms in the action. In an off-shell superfield formulation, though, the supersymmetry transformations are independent of the form of the action. In this case, it only remains to list all the supersymmetry invariants with a given number of derivatives. A prescription for generating all possible such terms might only exist if the superfields are unconstrained. The unconstrained superfield formulation of $N = 1$ supersymmetry is familiar (see e.g., [5]), while harmonic superspace [6] gives such an unconstrained formulation for $N = 2$, and 3 supersymmetries (see e.g., [7]).[1]

Superspace Chern-Simons-like terms are gauge-invariant terms in the action which cannot be written solely in terms of the field-strength superfield and derivatives, but must also include at least one vector potential superfield. For example, in the $N = 1$ superspace description of a U(1) gauge theory, the vector potential superfield is the real, gauge-variant, field $V$, while the field strength superfield is the chiral $W_\alpha = -\frac{1}{4}\overline{D}^2 D_\alpha V$. The question of the existence of superspace Chern-Simons-like terms in this theory is then whether there are gauge-invariant terms of the form $\int d^4x d^4\theta\, f(V, D_\alpha, \overline{D}_{\dot\alpha})$ which cannot be rewritten as $\int d^4x d^4\theta\, g(W_\alpha, \overline{W}_{\dot\alpha}, D_\alpha, \overline{D}_{\dot\alpha})$ by integration by parts in $x$ or $\theta$ (or similarly for chiral terms integrated over only half of superspace).

Superspace Chern-Simons-like terms are a logically broader category than Chern-Simon-like terms: expansion of a superspace Chern-Simons-like term in component fields need not give rise to a Chern-Simons-like term for the component gauge fields. Indeed, as mentioned above, it is believed that such Chern-Simons-like terms do not exist in even space-time dimensions. On the other hand, superspace Chern-Simons-like terms are, in fact, known to exist; the 2-derivative (kinetic) terms of the $N = 3$ supersymmetric U(1) theory [15] are given as superspace Chern-Simons terms in $N = 3$ harmonic superspace [16, 15, 17, 7].

[1] For the $N = 2$ vector multiplet, which will be the focus of this paper, other unconstrained superfield formalisms exist: $N = 2$ global superspace [8] with unconstrained real potential superfield [9, 10] of derivative weight $w(V_{(ij)}) = -3$ related to the field strength by $W = \overline{D}^4 D^i \cdot D^j V_{ij}$; or projective superspace [11, 12] with unconstrained analytic potential superfield [13] of derivative weight $w(V) = -1$ related to the field strength by $4\overline{W} = \oint d\zeta \triangle^2 V$. Both these formalisms suffer from the same problem of negative derivative weight potential superfields as the harmonic superspace formalism. It is algbraically more complicated to search for Chern-Simons-like terms in the global superspace formalism because of the potential's more negative weight. The projective and harmonic formalisms turn out to be equivalent in their algebraic complexity [14].



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In this paper we will initiate a search for such superspace Chern-Simons-like terms in supersymmetric theories. By the scaling argument discussed above, this is a logically pressing issue for making a systematic derivative expansion on the Coulomb branch of $N = 2$ theories. It is more a matter of curiosity whether such terms exist in $N = 1$ superspace, so we will only comment briefly on the $N = 1$ case in what follows. In either case, the existence of superspace Chern-Simons-like terms is a difficult algebraic question.

There are two broad strategies we pursue to search for superspace Chern-Simons-like terms. We can use

- gauge-variant (vector potential) superfields, or
- gauge-invariant (field strength) component fields.

Each of these strategies has its limitations which we now describe.

The gauge-variant superfield strategy, pursued in section 3 below, is the straightforward search for gauge invariant terms in the action involving vector multiplets of the form (schematically)

$$S = \int d\zeta\, f(V, D)\,, \tag{2.1}$$

which cannot be rewritten in the form

$$S = \int d\zeta\, g(W, D)\,. \tag{2.2}$$

Here $d\zeta$ is the appropriate superspace measure, $D$ denotes all the various superspace covariant derivatives, $V$ denotes the potential superfield, and $W$ the field strength superfield. For instance, for $N = 1$ supersymmetry, $V$ is a real scalar superfield, and $W_\alpha$ is a chiral spinor superfield, while for $N = 2$ in harmonic superspace $V^{++}$ is a real analytic scalar superfield, while $W$ is a chiral scalar superfield. (The following arguments work for both $N = 1$ and $N = 2$ supersymmetry, so we drop the indices on $V$ and $W$.) The problem with this strategy is that at a given order in the derivative expansion an arbitrary number of $V$'s can enter since they have non-positive derivative weight, $w(V) \leq 0$. To make progress, we then organize our search by looking for superspace Chern-Simons-like terms that can be written with only a set number, $\ell$, of $V$'s, schematically:

$$S = \int d\zeta\, V^\ell F(W, D)\,. \tag{2.3}$$

In section 3 we carry this out for $\ell = 1$. At $\ell = 2$ such a direct search is already algebraically prohibitively complicated. We will find, though, an interesting $\ell = 1$ term which is Chern-Simons-like *globally* on the Coulomb branch of $N = 2$ theories.

The second strategy makes the simplifying assumption that there are no Chern-Simons-like terms (as opposed to *superspace* Chern-Simons-like terms) in even dimensions, so that in components, every term can be written in terms of field-strengths $F_{\mu\nu}$ without any explicit gauge potentials $A_\mu$. Indeed, a partial fixing of the gauge invariance for either $N = 1$ or $N = 2$ vector multiplets allows us to set all but a finite number of auxiliary fields to zero, leaving the gauge-variant vector potential, $A_\mu$, as well as gauge invariant scalars



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and spinors, which we'll collectively denote by $\phi$, as component fields. In this gauge the general term in the action (2.1) becomes

$$S = \int d^4x\, g(A_\mu, \phi, \partial_\nu)\,, \tag{2.4}$$

where $g$ is Lorentz invariant and gauge invariant under $\delta A_\mu = \partial_\mu \ell$. Since the $\phi$'s are gauge invariant and assuming that there are no Chern-Simons-like terms in even dimensions, it follows that up to total derivatives (2.4) can be written as

$$S = \int d^4x\, h(F_{\mu\nu}, \phi, \partial_\rho)\,. \tag{2.5}$$

Thus we can search for superspace Chern-Simons-like terms making no assumptions on the number of factors, $\ell$, of $V$ that appear by abandoning superfields and working in terms of gauge-invariant component fields. We carry this out in section 4 to show that there are no 3-derivative superspace Chern-Simons-like terms on a one-dimensional $N = 2$ Coulomb branch. The price we pay is that since we are working in components, we have to check supersymmetry "by hand".

It may be helpful at this point to remark on a connection between such superspace Chern-Simons-like terms and the issue of locality in the Grassmann coordinates of superspace. Since $F_{\mu\nu}$, $\phi$, and their derivatives are just components of the field strength superfield $W$ and its derivatives, we can write (2.5) as

$$S = \int d^4x\, h\left(\textstyle\int d\theta_1 j_1(W,D)\,,\, \int d\theta_2 j_2(W,D)\,, \ldots\right), \tag{2.6}$$

where the $j_n$ are arbitrary functions of superspace covariant derivatives and $W$'s, and the $d\theta_i$ are appropriate integration measures over the Grassmann-odd superspace coordinates. Thus we have rewritten the general vector multiplet term (2.1) solely in terms of the field strength superfield. But (2.6) is not local in superspace. Such a superspace-local term would have just a single integral over the Grassmann-odd coordinates, $S_{\rm local} = \int d^4x\, d\theta\, h(W,D)$. Fayet-Iliopoulos terms provide a simple example. Fayet-Iliopoulos terms are superspace Chern-Simons-like terms, in the sense that they cannot be written in terms of field strength superfields integrated over the usual superspace. For example, the $N = 1$ Fayet-Iliopoulos terms $\int d^4x d^4\theta\, V$ is gauge-invariant and cannot be written in terms of $W_\alpha$ integrated over the full superspace $\int d^4x d^4\theta$ or chiral superspace $\int d^4x_C d^2\theta$. However, by the above argument it can be written in terms of $W_\alpha$ integrated over some other superspace. Indeed, it is given by an integral over one quarter of superspace, $\int d^4x d\theta^\alpha\, W_\alpha$, and is supersymmetric by virtue of the Bianchi identity that $W_\alpha$ satisfies. Unlike (2.6) it is local in superspace, but only because it involved only a single field. See [4] for a more detailed discussion of superspace locality.

A third strategy for finding superspace Chern-Simons-like terms, which we hope to report on elsewhere [18], is to use an instanton calculation to show the existence of certain component terms in the action which are known not to arise from any supersymmetric term involving just field strength superfields. The main drawback of this strategy is that it is not systematic, so cannot rule out the existence of general Chern-Simons-like terms, but only of certain very special ones.



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## 3. Gauge-variant superfield arguments

In the rest of this paper we work on the Coulomb branch of an $N = 2$ gauge theory where the low energy effective action at a generic vacuum includes only massless U(1) vector multiplets and massless neutral hypermultiplets. Furthermore, for simplicity we will ignore the hypermultiplets, and only consider terms with vector superfields. The propagating component fields of a U(1) vector superfield are massless neutral scalars and spinors, $\phi$, $\psi_\alpha$, and U(1) vectors, $A^\mu$. In harmonic superspace the vectormultiplet is represented either by the potential superfield $V^{++}$ or the field strength superfield $W$.

**Harmonic superspace:** We now very briefly review the salient points of the harmonic superspace formalism concerning vector superfields. We follow the notation of [7] where a detailed exposition of harmonic superspace can be found; a concise review appears in [4].

An important feature of harmonic superspace is that, in addition to the usual space-time directions described by coordinates $x^\mu$ and Grassmann-odd directions with spinor coordinates, $\theta^\pm_\alpha$ and $\overline{\theta}^\pm_{\dot\alpha}$, there is also a 2-sphere described by commuting harmonic coordinates $u^\pm_i$, $i \in \{1, 2\}$. Expansion in the $u$'s gives rise to an infinite number of auxiliary fields. Though terms in the effective action need not be local in the $u$'s, there exists a systematic procedure to list all such terms [4]. In the case of vector superfields, we will see that the $u$'s play only a minor role.

Superspace covariant derivatives, $D^\pm_\alpha$ and $\overline{D}^\pm_{\dot\alpha}$, are introduced in the usual way, along with a set of covariant $u$ derivatives denoted $D^{++}$, $D^{--}$, and $D^0$. The $u$-derivatives satisfy an $\mathrm{SU}(2)_R$ algebra

$$[D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}\,, \qquad [D^{++}, D^{--}] = D^0\,, \tag{3.1}$$

while the covariant derivatives satisfy the $N = 2$ algebra

$$\{D^\pm_\alpha, \overline{D}^\mp_{\dot\alpha}\} = \mp 2i\,\partial\!\!\!/_{\alpha\dot\alpha}\,, \qquad [D^{\pm\pm}, D^\mp_\alpha] = D^\pm_\alpha\,, \qquad [D^{\pm\pm}, \overline{D}^\mp_{\dot\alpha}] = \overline{D}^\pm_{\dot\alpha}\,, \tag{3.2}$$

with all other (anti)commutators vanishing. Eqs. (3.2) and (3.1) give the form of the $N = 2$ algebra on harmonic superspace that we will use. $N = 2$ supersymmetry invariants can be formed by integrating a general harmonic superfield over all the superspace coordinates with measure $\int du\, d^4x\, d^4\theta^+\, d^4\theta^-$, where $du$ is the appropriate measure for integration over the $u$-sphere.

The $\pm$ superscripts denote the charge under $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$. $N = 2$ invariant terms are required to be neutral under this U(1). Also, all functions of the $u^\pm$ are required to be harmonic, which is to say that they have regular power series expansions in the $u^\pm$.

Two different constraints, the *chiral constraint* and the *analytic constraint*, can be used to reduce superfield representations in $N = 2$ harmonic superspace. The chiral constraint,

$$\overline{D}^+_{\dot\alpha} W = \overline{D}^-_{\dot\alpha} W = 0\,, \tag{3.3}$$

is solved by introducing a chiral space-time coordinate $x^C$ annihilated by $\overline{D}^\pm$. Then the chiral constraint is solved by an arbitrary (unconstrained) superfield independent of the



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$\overline{\theta}^{\pm}$'s: $W = W(x_C^\mu, \theta_\alpha^\pm, u_i^\pm)$. The field-strength superfield for the vector multiplet is such a chiral superfields. Supersymmetry invariants can be constructed by integrating chiral superfields against the measure $\int du\, d^4x_C\, d^4\theta = \int du\, d^4x\, D^4$, where $D^4 \equiv \frac{1}{16}(D^+)^2(D^-)^2$. The analytic constraint,

$$D_\alpha^+ V = \overline{D}_{\dot\alpha}^+ V = 0\,, \tag{3.4}$$

is solved by introducing an analytic space-time coordinate $x_A$ annihilated by $D^+$ and $\overline{D}^+$, so that an arbitrary (unconstrained) superfield independent of $\theta^-$ and $\overline{\theta}^-$, $V = V(x_A^\mu, \theta_\alpha^+, \overline{\theta}_{\dot\alpha}^+, u_i^\pm)$, solves the analytic constraint. These *analytic superfields* are useful for describing the vector potential superfield. Supersymmetry invariants can be constructed by integrating analytic superfields against the measure $\int du\, d^4x_A\, d^4\theta^+ = \int du\, d^4x\, (D^-)^2(\overline{D}^-)^2$. Note, in particular, that $d^4\theta^+$ has $\mathrm{U}(1)_R$ charge $-4$ because Grassmann integration is differentiation.

The unconstrained $N=2$ vector multiplet superfield is a (real) analytic (3.4) superfield $V^{++}$ transforming under U(1) gauge transformations as

$$\delta V^{++} = -D^{++}\lambda\,, \tag{3.5}$$

where $\lambda$ is an arbitrary real analytic superfield. The gauge invariant field strength superfield is constructed as follows. First, another gauge potential superfield $V^{--}$ is defined in terms of $V^{++}$ as the solution to the differential equation in $u^\pm$

$$D^{++}V^{--} = D^{--}V^{++}\,, \tag{3.6}$$

which has a unique solution by virtue of the harmonicity requirement on the $u$-sphere. $V^{--}$ is not an analytic (or anti-analytic) superfield, but is real and transforms under gauge transformations as $\delta V^{--} = -D^{--}\lambda$. Two useful identities involving $V^{--}$ are

$$D_\alpha^- V^{++} = -D^{++}D_\alpha^+ V^{--}\,, \qquad D_\alpha^- V^{--} = -D^{--}D_\alpha^+ V^{--}\,, \tag{3.7}$$

and similarly with $\overline{D}$'s. The field strength superfield is then defined by

$$W = -\frac{1}{4}(\overline{D}^+)^2 V^{--}\,. \tag{3.8}$$

It is a straight forward exercise, using the $N=2$ algebra (3.1) and (3.2), to check that $W$ is gauge invariant, chiral (3.3), $u$-independent

$$D^{\pm\pm}W = 0\,, \tag{3.9}$$

and satisfies the Bianchi identities

$$D^\pm \cdot D^\pm W = \overline{D}^\pm \cdot \overline{D}^\pm \overline{W}\,, \qquad D^\pm \cdot D^\mp W = \overline{D}^\pm \cdot \overline{D}^\mp \overline{W}\,. \tag{3.10}$$

The $u$-independence of $W$ implies that in expressions involving the field strength superfields alone (i.e., no $V^{\pm\pm}$'s), the integration over the auxiliary $u$-sphere can be done separately, leaving an expression in standard $N=2$ superspace with coordinates $\{x^\mu, \theta_\alpha^\pm, \overline{\theta}_{\dot\alpha}^\pm\}$.

The lowest component of $W$ is the complex scalar $\phi$ whose vevs parameterize the Coulomb branch. Thus $W$ must be assigned derivative weight $w(W)=0$. Since $w(D) = 1/2$ and the $u$'s and therefore the $D^{\pm\pm}$ derivatives have weight 0, (3.8) and (3.6) imply that $w(V^{\pm\pm}) = -1$. This negative weight is the source of the problem of Chern-Simons-like terms in $N=2$ effective actions.



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**Terms with no $V$s:** Before starting our search for harmonic superspace Chern-Simons-like terms, we first review the classification of Coulomb branch terms that can be written solely using the field strength superfield and its derivatives. The complete set of such terms up to four derivatives is [4]

$$\begin{aligned}
S_1 &= \int d^4x\, d\theta^i \cdot d\theta^j\, \xi_{ij}\, W\,, \qquad i,j \in \{1,2\}\,, \quad \xi_{ij} \in \mathbb{R}\,,\\
S_2 &= \int d^4x\, d^4\theta\ \mathcal{F}(W) + \text{c.c.}\,,\\
S_{4a} &= \int d^4x\, d^4\theta\ \partial_\mu W \partial^\mu W\, \mathcal{G}(W) + \text{c.c.}\,,\\
S_{4b} &= \int d^4x\, d^4\theta\, d^4\overline{\theta}\, \mathcal{H}(W,\overline{W})\,.
\end{aligned} \tag{3.11}$$

The 1-derivative term is the Fayet-Iliopoulos term; though an integral over only 1/4 of superspace, it is $N=2$ invariant by virtue of the extra constraint (3.10) that $W$ satisfies. The 2-derivative term is the well-known holomorphic prepotential term, encoding generalized kinetic, Yukawa, and $\psi^4$ terms. There are no 3-derivative terms, and two independent 4-derivative terms, the first of which is holomorphic; note that when there is only a single vector multiplet, $S_{4a}$ can be rewritten using the Bianchi identity as an $S_{4b}$ term [4]. These terms will be discussed in more detail with a view towards possible global obstructions to their existence in section 5 below.

Now we turn to the question of whether there exist gauge invariant U(1) vector multiplet terms which cannot be written solely in terms of the field strength multiplets $W$. Let us examine this possibility by assuming that such a term can be written with just one power of the potential superfield, i.e., schematically of the form

$$S = \int d\zeta\, V^{\pm\pm}\, f(W,D)\,, \tag{3.12}$$

where $d\zeta$ is some superspace measure and $f$ is an arbitrary function of field strength superfields and covariant derivatives. (Note that, by integration by parts, we can always write such terms with no derivatives acting on $V$.) We must first determine the conditions on $f$ such that $S$ is gauge invariant. Then we must show that it cannot be written as one of the terms in (3.11). Only then will we have found a superspace Chern-Simons-like term.

There are two possible choices for the measure $d\zeta$ in (3.12). Since $V^{++}$ is analytic, $S$ could be $N=2$ invariant if $d\zeta = dud^4xd^4\theta^+$, the integration over the analytic half of superspace. The other possibility is that $d\zeta$ could be the integration measure over all of superspace. We will explore these two possibilities in turn.

**One-derivative terms with one $V$:** If the integration is only over analytic superspace, the integrand in (3.12) must be analytic superfield. This limits its form to

$$S_A = \int d^4xdud^4\theta^+\, f_a^{++} {V^{++}}_a \tag{3.13}$$

where $a$ is an index labelling different U(1) vector multiplets, and $f^{++} = f^{++}(u^\pm, D^{\pm\pm}, \partial_\mu)$ is an arbitrary function of $u$ and $u$- and $x$-derivatives; in particular, no dependence on $W$



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or $\overline{W}$ is allowed by analyticity. Since the derivatives can be taken to not act on $V^{++}$ by integration by parts, we can drop the derivatives altogether, and $f^{++} = f^{++}(u^\pm)$. Thus $S_A$ can only be a 1-derivative term since $w(V^{++}) = -1$ and $w(d^4\theta^+) = +2$. The gauge variation of $S_A$ is then

$$\delta S_A = -\int d^4x du d^4\theta^+ \, f^{++}(u) \cdot D^{++}\lambda = \int d^4x du d^4\theta^+ \, \left[D^{++}f^{++}(u)\right] \cdot \lambda \tag{3.14}$$

where in the second step we have integrated by parts, and where we have suppressed the $a$ index, using $\cdot$ to denote contraction over these flavor indices. Since $\lambda$ are arbitrary analytic superfields, gauge invariance implies that $D^{++}f^{++} = 0$, or that $f^{++}$ is independent of $u^-$. In order to have total $\mathrm{U}(1)_R$ charge $+2$, we therefore have $f^{++}(u) = u_i^+ u_j^+ \xi^{ij}$, where $\xi^{ij}$ are some constants. Now, $d^4\theta^+ = d^2\theta^+ (\overline{D}^-)^2$, so

$$S_A = \int d^4x du d^2\theta^+ \, u_i^+ u_j^+ \xi^{ij} (\overline{D}^-)^2 V^{++} \,. \tag{3.15}$$

From (3.7), (3.2), and (3.8) it follows that $(\overline{D}^-)^2 V^{++} = -D^{++}\overline{D}^- \cdot \overline{D}^+ V^{--} - 4W$. Inserting this into (3.15), the first term vanishes after integration by parts, leaving $S_A \propto \int d^4x d\theta_i \cdot d\theta_j \sigma^{ij} W$, which is just the Fayet-Iliopoulos term $S_1$ in (3.11).

We now turn to the terms written as integrals over the full superspace. Since $w(d^8\theta) = 4$ and $w(V) = -1$, these terms have at least 3 derivatives. We will examine only the 3- and 4-derivative terms found this way.

**Three-derivative terms with one $V$:** The general 3-derivative term is

$$S_3 = \int d^4x du d^8\theta \left\{ g^{(-2)}(W, \overline{W}, u) \cdot V^{++} + g^{(+2)}(W, \overline{W}, u) \cdot V^{--} \right\} . \tag{3.16}$$

Now, any function $g^{(+2)}$ can be written as $g^{(+2)} = D^{++}g^{(0)}$ for some $g^{(0)}$ by harmonicity in $u$. Then

$$\begin{aligned} \int du \, g^{(+2)} V^{--} &= \int du \, (D^{++}g^{(0)}) V^{--} = -\int du \, g^{(0)} D^{++} V^{--} = -\int du \, g^{(0)} D^{--} V^{++} \\ &= \int du \, (D^{--}g^{(0)}) V^{++} = \int du \, \tilde{g}^{(-2)} V^{++} \,. \end{aligned}$$

Thus the $V^{--}$ term can be converted to the $V^{++}$ term, and so can be dropped from (3.16). The gauge variation of $S_3$ after integration by parts is

$$\delta S_3 = \int d^4x du d^8\theta \left[ D^{++} g^{(-2)} \right] \cdot \lambda \,, \tag{3.17}$$

which vanishes if and only if

$$D^{++} g^{(-2)} = D^+ \overline{h}^{(-1)} + \overline{D}^+ h^{(-1)} \tag{3.18}$$

for some $\overline{h}^{(-1)}$ and $h^{(-1)}$, since $\lambda$ is analytic (i.e., annihilated by $D^+$ and $\overline{D}^+$). Since the left hand side of (3.18) is a function of $W$ and $\overline{W}$ only, we must have $\overline{h}^{(-1)} = \overline{f}(\overline{W}, u) D^+ V^{--}$



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and $h^{(-1)} = f(W,u)\overline{D}^+ V^{--}$ for some $f$ and $\overline{f}$. That then implies that $D^{++}g^{(-2)}$, and therefore $g^{(-2)}$, is a sum of a function of $W$ alone and of $\overline{W}$ alone. Thus, for gauge invariance, we have

$$S_3 = \int d^4x du d^8\theta \, V^{++} \cdot \left\{ g'^{(-2)}(W,u) + \overline{g}'^{(-2)}(\overline{W},u) \right\}. \tag{3.19}$$

But $\int d^8\theta = \int d^4\theta^+ (D^+)^2 (\overline{D}^+)^2$, and since both $D^+$ and $\overline{D}^+$ annihilate $V^{++}$ (by analyticity), and one or the other of them annihilate $g'$ or $\overline{g}'$ (by chirality of $W$ and $\overline{W}$), $S_3$ vanishes. Thus there are no 3-derivative terms with a single $V$.

**Four-derivative terms with one $V$:** The most general such expression has many terms:

$$\begin{aligned} S_4 = \int d^4x du d^8\theta \Big\{ & V^{++}{}_a \Big[ g_a^{(0)bc} D^- W_b \cdot D^- W_c + \widetilde{g}_a^{(-2)bc} D^- W_b \cdot D^+ W_c + \\ & \qquad + \widetilde{g}_a^{(-4)bc} D^+ W_b \cdot D^+ W_c + f_a^{(0)b} (D^-)^2 W_b + \\ & \qquad + f_a^{(-2)b} (D^- D^+) W_b + f_a^{(-4)b} (D^+)^2 W_b \Big] + \\ & + V^{--}{}_a \Big[ h_a^{(0)bc} D^+ W_b \cdot D^+ W_c + h_a^{(2)bc} D^- W_b \cdot D^+ W_c + \\ & \qquad + h_a^{(4)bc} D^- W_b \cdot D^- W_c + d_a^{(0)b} (D^+)^2 W_b + \\ & \qquad + d_a^{(2)b} (D^- D^+) W_b + d_a^{(4)b} (D^-)^2 W_b \Big] \Big\} + \text{c.c.} \end{aligned} \tag{3.20}$$

where the $d$, $f$, $g$, and $h$'s are all functions of $W$, $\overline{W}$, and the $u$'s. Even before requiring gauge invariance, this can be drastically simplified. First, consider the $V^{--}$ terms. We can write $h^{(4)} = D^{++}\widetilde{h}^{(2)}$, $h^{(2)} = D^{++}\widetilde{h}^{(0)}$, and $h^{(0)} = h + D^{++}\widetilde{h}^{(-2)}$ for some $\widetilde{h}$'s, and similarly for the $d$'s, where $h$ is the $u$-independent piece of $h^{(0)}$. Then integrate by parts on $D^{++}$, rewrite $D^{++}V^{--} \to D^{--}V^{++}$, and integrate by parts on $D^{--}$, to convert the $\widetilde{h}$ and $\widetilde{d}$ terms to terms of the same form as the $V^{++}$ terms. Therefore these terms can all be dropped. Next consider the $f^{(-2)}$ and $f^{(-4)}$ terms. By redefining $\widetilde{g} \to g$ by adding appropriate derivatives of the $f$'s with respect to $W$, these $f$ terms can be written as total $D^+$ derivatives times $V^{++}$. Integrating by parts on $D^+$ gives 0 since $D^+V^{++} = 0$ by analyticity. Therefore we can drop these terms as well. Thus $S_4$ has been simplified to

$$\begin{aligned} S_4 = \int d^4x du d^8\theta \Big\{ & V^{++} \Big[ g^{(0)} D^- W \vee D^- W + g^{(-2)} D^- W \vee D^+ W + \\ & \qquad + g^{(-4)} D^+ W \vee D^+ W + e^{(-2)} D^- W \wedge D^+ W + f^{(0)} (D^-)^2 W \Big] + \\ & + V^{--} \Big[ h \, D^+ W \vee D^+ W + d \, (D^+)^2 W \Big] \Big\} + \text{c.c.} \end{aligned} \tag{3.21}$$

where $e^{(-2)}$, $f^{(0)}$, and the $g^{(n)}$ are functions of $W$, $\overline{W}$, and the $u$'s; $d$ and $h$ are functions of $W$ and $\overline{W}$ only; we have suppressed the flavor indices, which should all be contracted with indices on the coefficient functions; we have introduced the notations $A \wedge B = \frac{1}{2}(A_a B_b - A_b B_a)$ and $A \vee B = \frac{1}{2}(A_a B_b + A_b B_a)$ for antisymmetrized and symmetrized indices respectively; and $e^{(-2)}$ and $g^{(-2)}$ are introduced as the antisymmetric and symmetric parts of $\widetilde{g}^{(-2)}$.

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We now demand that $S_4$ be gauge invariant. Taking the gauge variation, integrating by parts on $D^{\pm\pm}$ and on $D^+$ for the resulting $(D^-D^+)W$ term, and collecting terms gives, after some algebra,

$$\begin{aligned}\delta S_4 = \int d^4x du d^8\theta\, \lambda\Big\{&[D^{++}g^{(0)}]D^-W \vee D^-W + [D^{++}f^{(0)}](D^-)^2W + \\ &+[D^{++}e^{(-2)} - 2\partial \wedge (d + f^{(0)})]D^-W \wedge D^+W + \\ &+[D^{++}g^{(-2)} - 2\partial \vee (d + f^{(0)}) + 2(h + g^{(0)})]D^-W \vee D^+W + \\ &+[D^{++}g^{(-4)} + g^{(-2)}]D^+W \vee D^+W\Big\} + \text{c.c.}\end{aligned} \tag{3.22}$$

where $\partial = \partial/\partial W$. Gauge invariance then implies that the terms in square brackets must vanish, giving

$$\begin{aligned} D^{++}g^{(0)} &= D^{++}f^{(0)} = 0\,, \\ D^{++}e^{(-2)} &= 2\partial \wedge (d + f^{(0)})\,, \\ D^{++}g^{(-2)} &= 2\partial \vee (d + f^{(0)}) + 2(h + g^{(0)})\,, \\ D^{++}g^{(-4)} &= -g^{(-2)}\,. \end{aligned} \tag{3.23}$$

The first line of (3.23) implies that $g^{(0)} = g$ and $f^{(0)} = f$ are independent of $u$. Thus the right hand sides of the second and third lines are $u$-independent, which then implies that $e^{(-2)} = g^{(-2)} = 0$, which in turn implies $g^{(-4)} = 0$ by the fourth line. Furthermore, the right hand sides of the second and third lines then vanish, giving $\partial \wedge (d + f) = 0$, and $\partial \vee (d + f) = g + h$. Define $\hat{g} = g - \partial \vee f$ and $\hat{h} = h - \partial \vee d$ so that

$$\begin{aligned} S_4 = \int d^4x du d^8\theta \Big\{&V^{++}[\hat{g}\, D^-W \vee D^-W + D^-(f\, D^-W)] + \\ &+ V^{--}[\hat{h}\, D^+W \vee D^+W + D^+(d\, D^+W)]\Big\} + \text{c.c.}\end{aligned} \tag{3.24}$$

where $d$, $f$, $\hat{g}$, and $\hat{h}$ are $u$-independent functions of $W$ and $\overline{W}$ satisfying

$$\partial \wedge (d + f) = 0\,, \qquad \text{and} \qquad \hat{g} + \hat{h} = 0\,. \tag{3.25}$$

This can be simplified further. Consider the following manipulation of the $\hat{g}$ term:

$$\begin{aligned}\int d^4x du d^8\theta\, V^{++}\, \hat{g}\, D^-W \vee D^-W &= \int d^4x du d^8\theta\, V^{++}\, \hat{g}\, (D^{--}D^+W) \vee D^-W \\ &= \int d^4x du d^8\theta\, V^{++}\, \hat{g}\, D^{--}(D^+W \vee D^-W) \\ &= -\int d^4x du d^8\theta\, (D^{--}V^{++})\, \hat{g}\, D^+W \vee D^-W \\ &= -\int d^4x du d^8\theta\, (D^{++}V^{--})\, \hat{g}\, D^+W \vee D^-W \\ &= \int d^4x du d^8\theta\, V^{--}\, \hat{g}\, D^{++}(D^+W \vee D^-W) \\ &= \int d^4x du d^8\theta\, V^{--}\, \hat{g}\, D^+W \vee (D^{++}D^-W) \\ &= \int d^4x du d^8\theta\, V^{--}\, \hat{g}\, D^+W \vee D^+W\,,\end{aligned} \tag{3.26}$$



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where we have used extensively that $W$ is $u$-independent. This shows that the $\hat{g}$ term is the same as the $\hat{h}$ term. But since $\hat{g} = -\hat{h}$ by (3.25), they cancel. Now consider the following manipulation of the $f$ term:

$$\begin{aligned}
\int d^4x du d^8\theta \, V^{++} \, D^-(f \, D^- W) &= -\int d^4x du d^8\theta \, (D^- V^{++}) \, f \, D^- W \\
&= \int d^4x du d^8\theta \, (D^{++} D^+ V^{--}) \, f \, D^- W \\
&= -\int d^4x du d^8\theta \, (D^+ V^{--}) \, f \, D^{++} D^- W \\
&= -\int d^4x du d^8\theta \, (D^+ V^{--}) \, f \, D^+ W \\
&= \int d^4x du d^8\theta \, V^{--} \, D^+(f \, D^+ W) \, , 
\end{aligned} \tag{3.27}$$

which is of the same form as the $d$ term. Calling $A(W, \overline{W}) \equiv d + f$, and restoring the flavor indices, the final form for gauge-invariant 4-derivative terms with one $V$ is

$$S_4 = \int d^4x du d^8\theta \, V_a^{--} \, D^+ \left( A_a^b(W, \overline{W}) D^+ W_b \right) + \text{c.c.} \, , \tag{3.28}$$

where, from (3.25), $A$ satisfies

$$\partial^c A_a^b - \partial^b A_a^c = 0 \, . \tag{3.29}$$

The next step is to determine when this term can be rewritten solely in terms of $W$'s and $\overline{W}$'s. To do this we need to have the $D^+$'s act on $V^{--}$. But (3.29) is precisely the local integrability condition for

$$A_a^b = \partial^b B_a \tag{3.30}$$

for some $B_a(W, \overline{W})$. In this case $A_a^b D^+ W_b = \partial^b B_a D^+ W_b = D^+ B_a$, and $S_4$ becomes

$$S_4 = \int d^4x du d^8\theta \, V_a^{--} \, (D^+)^2 B_a = \int d^4x du d^8\theta \, B_a \, (D^+)^2 V_a^{--} = -4 \int d^4x d^8\theta \, B_a \, \overline{W}^a \, , \tag{3.31}$$

written solely in terms of field strength superfields. However, this rewriting was possible only locally on the Coulomb branch, since globally there might be an obstruction to the integrability of (3.30). Thus (3.28) may, in fact, be a superspace Chern-Simons-like term, albeit only globally on the moduli space. We will return in section 5 to discuss the existence and implications of such terms.

Aside from discovering this new term, there are few general lessons we can extract from this calculation. Since it assumed a specific form, namely only one explicit power of $V^{\pm\pm}$, it allows no general statements to be made about the existence of Chern-Simons-like terms at a given derivative order. Because even with one power of $V$ the calculation was so algebraically complex, it seems unlikely that this strategy can be usefully extended to a general argument valid for all powers of $V$. Indeed, even at $V^2$ the algebra is prohibitively complicated. As a small example, consider the following 3-derivative term with two $V$'s:

$$S_3 = \int d^4x du d^8\theta \, \left[ f_{ab}(W, \overline{W}) + g_{ab}(\overline{W}, u) \right] \, D^+ V_a^{--} D^- V_b^{++} \, , \tag{3.32}$$



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where both $f$ and $g$ are symmetric on $a$ and $b$. We leave as an exercise for the masochistic reader to show, first, that this term is gauge invariant, and second, that it actually vanishes.

For these reasons, we now turn to a more systematic approach to the search for superspace Chern-Simons-like terms.

## 4. Gauge-invariant component arguments

We now search for superspace Chern-Simons-like terms by looking directly at the component expansion of the vector multiplet. As discussed in section 2, assuming that there are no Chern-Simons-like terms (as opposed to *superspace* Chern-Simons-like terms) in even dimensions, any gauge invariant term can be written in terms of the components of the field strength vector multiplet.

The vector multiplet contains scalar fields $\phi$ and $\overline{\phi}$, with derivative weight zero; a spinor field $\psi^\alpha$, with derivative weight 1/2; a triplet of auxiliary scalar fields $D^{++}, D^{--}$, and $D^{+-}$, with derivative weight one; and U(1) gauge field strengths $F^{(\alpha\beta)}$ and $\overline{F}^{(\dot\alpha\dot\beta)}$, symmetric on the spinor indices[2] also with derivative weight one. In addition to these fields, we may have spacetime derivatives, carrying derivative weight one. We write these derivatives contracted with a pauli matrix, giving them a dotted and an undotted index: $\partial\!\!\!/_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu$. These components are related to the field strength chiral superfield $W$ and its conjugate $\overline{W}$ by

$$
\begin{aligned}
\phi &= W|_{\theta=\overline{\theta}=0}\\
\overline{\phi} &= \overline{W}|_{\theta=\overline{\theta}=0}\\
\psi^\pm_\alpha &= D^\pm_\alpha W|_{\theta=\overline{\theta}=0}\\
\overline{\psi}^\pm_{\dot\alpha} &= \overline{D}^\pm_{\dot\alpha}\overline{W}|_{\theta=\overline{\theta}=0}\\
D^{\pm\pm} &= D^{\pm\alpha}D^\pm_\alpha W|_{\theta=\overline{\theta}=0} = \overline{D}^{\pm\pm}\\
D^{+-} &= D^{+\alpha}D^-_\alpha W|_{\theta=\overline{\theta}=0} = \overline{D}^{+-}\\
F_{(\alpha\beta)} &= (D^{+\alpha}D^{-\beta} + D^{+\beta}D^{-\alpha})W|_{\theta=\overline{\theta}=0}\\
\overline{F}_{(\dot\alpha\dot\beta)} &= (\overline{D}^{+\dot\alpha}\overline{D}^{-\dot\beta} + \overline{D}^{+\dot\beta}\overline{D}^{-\dot\alpha})\overline{W}|_{\theta=\overline{\theta}=0}\,.
\end{aligned}
\tag{4.1}
$$

Any expression written in terms of these fields will be automatically gauge invariant, but not manifestly $N = 2$ supersymmetric.

We can now organize the search for superspace Chern-Simons-like term order-by-order in the derivative expansion. The non-Chern-Simons-like terms with four or fewer derivatives were listed in section 3, (3.11). It is easy to see in components [8, 19] that all 1- and 2-derivative terms are included in this list. Since there are no 3-derivative terms in this list, if we find any in the component method, we will have then found a superspace Chern-Simons-like term.

We will now show that there are no gauge invariant $N = 2$ supersymmetric 3-derivative terms with just a single vector multiplet. We look at only one U(1) vector multiplet for simplicity. We will comment on the extension to many U(1)'s below.

[2] Or, equivalently, anti-symmetric on spacetime indices.



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Our strategy for showing that there are no 3-derivative terms is to look at a possible term in the action, find its supersymmetry variation, and then show that this variation cannot be cancelled. Once this term is shown not to contribute, we then look at another action term, and so on, until all are exhausted. The key to doing this efficiently is to eliminate the terms in a particular order. We have found that it is convenient to organize the terms by decreasing number of U(1) field strength fields $F$. This is both because the $F$ fields can only be obtained in one way in a supersymmetry variation, and also because of the limited number of ways in which they can be included, due to their two spinor indices and Lorentz invariance.

Note that an action will be supersymmetric not only if the variation of the lagrangian vanishes, but also if it is just a total derivative. So, it is possible that the variations of combinations of terms do not cancel, but add to form a total derivative. In order for this to happen, the terms must all have the same fields and Lorentz structure, but with derivatives acting on different fields. Since we will almost never have to keep track of where derivatives act,[3] when we say that terms cannot cancel, then they also cannot add to form a total derivative.

$N=2$ supersymmetry is preserved when the action is invariant under four independent supersymmetry transformations generated by

$$D^+_\alpha,\ D^-_\alpha,\ \overline{D}^+_{\dot\alpha},\ \overline{D}^-_{\dot\alpha}\,. \tag{4.2}$$

For our purposes checking the supersymmetry variation under one of these will be equivalent to checking the variation with respect to the others. For definiteness we usually look at the $D^+_\alpha$ variation, and unless otherwise specified this will be what we mean by *the* supersymmetry variation.

The supersymmetry transformations of the component fields are

$$\begin{aligned}
D^\pm_\alpha \phi &= \psi^\pm_\alpha \\
D^\pm_\alpha \overline{\phi} &= 0 \qquad\qquad (4.3)\\
D^\pm_\alpha \psi^\pm_\beta &= \frac{1}{2}\epsilon_{\alpha\beta} D^{\pm\pm} \\
D^\pm_\alpha \psi^\mp_\beta &= \pm F_{(\alpha\beta)} + \frac{1}{2}\epsilon_{\alpha\beta} D^{+-} \\
D^\pm_\alpha \overline{\psi}^\pm_{\dot\beta} &= 0 \\
D^\pm_\alpha \overline{\psi}^\mp_{\dot\beta} &= \mp 2i\, \partial\!\!\!/_{\alpha\dot\beta}\overline{\phi} \qquad\qquad (4.4)\\
D^\pm_\alpha D^{\pm\pm} &= 0 \\
D^\pm_\alpha D^{\mp\mp} &= \mp 4i\, (\partial\!\!\!/\overline{\psi}^\mp)_\alpha \\
D^\pm_\alpha D^{+-} &= \mp 2i\, (\partial\!\!\!/\overline{\psi}^\pm)_\alpha \qquad\qquad (4.5)\\
D^\pm_\alpha F_{(\beta\gamma)} &= 4i\, \epsilon_{\alpha(\beta}(\partial\!\!\!/\overline{\psi}^\pm)_{\gamma)} \\
D^\pm_\alpha \overline{F}_{(\dot\beta\dot\gamma)} &= \mp 2i\, \partial\!\!\!/_{\alpha(\dot\beta}\overline{\psi}^\pm_{\dot\gamma)} \qquad\qquad (4.6)
\end{aligned}$$

as is easily read off from the superfield expressions (4.1) and the $N=2$ algebra (3.2).

[3]Though we do need to keep track of where derivatives act when looking at terms with one $F$ and no fermions, but only for the purpose of finding out what fields we need in such a term. The possibility of a total derivative does not enter there.



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**Terms with three $F$'s:** First of all, we note that with only one (or two) distinct vector multiplets, we cannot have a term in the action with three $F$ fields, by Lorentz invariance and symmetry on spinor indices. Since each $F$ has derivative weight one, there can be no spinors $\psi$ in a 3-$F$, 3-derivative term, and so the three $F$s' spinor indices must be contracted. Because the $F$'s are symmetric on their spinor indices (and because spinor index contraction is antisymmetric) this trace is necessarily equal to its negative and therefore zero. Similarly there can be no action terms with three $\overline{F}$'s, and we cannot make a derivative weight three Lorentz scalar that contains both $F$ and $\overline{F}$.

**Terms with two $F$'s:** Now we look at action terms with two $F$ fields. The only possible 3-derivative Lorentz scalars are

$$(\psi^{\pm})^2\,\mathrm{tr}(F^2)\,, \qquad (\overline{\psi}^{\pm})^2\,\mathrm{tr}(F^2)\,, \qquad D^{\pm\pm}\,\mathrm{tr}(F^2)\,, \tag{4.7}$$

as well as their conjugates, involving $\overline{F}^2$. These terms can have arbitrary coefficients which are functions of the scalars $\phi$ and $\overline{\phi}$. These scalar function coefficients are suppressed below since they will play no part in our proof, but they should be considered to be present in any term. The traces in (4.7) mean contraction on spinor indices. There are other such terms, but they can always be written as one of the above using a Fierz identity. Each of these terms has a definite $\mathrm{U}(1)_R$ charge in $N = 2$ harmonic superspace, and only terms with the same charge can have variations that cancel.

The first term above, $(\psi^{\pm})^2\,tr(F^2)$, must have either a $D^+_\alpha$ or $D^-_\alpha$ variation that gives an additional $F$, resulting in a 3-$F$ variation term. This term does not vanish, since the result is not a Lorentz scalar and does not have the $F$'s traced. Since we have no 3-$F$ action terms, and we cannot get two $F$'s from a single variation, this variation term must be cancelled by the variation of another 2-$F$ term. Since there is only one term with two $F$'s plus fermions for each $\mathrm{U}(1)_R$ charge, there is no other term to cancel this variation. A similar argument holds for the second term above, based on the nonexistence of a term with both an $F$ and an $\overline{F}$.

To show that the $D^{\pm\pm}\,tr(F^2)$ term cannot appear in the action we instead look at the term $D^{\pm\pm}\,tr(\overline{F}^2)$ for simplicity.[4] For the terms with $\mathrm{U}(1)_R$ charge 0 or -2 we can act with $D^+_\alpha$ on the $D^{\pm\pm}$ to get a term with two $\overline{F}$'s and a $\overline{\psi}$:

$$\mathrm{tr}(\overline{F}^2)(\partial\!\!\!/\overline{\psi}^{\pm})_\alpha. \tag{4.8}$$

(For the term with $\mathrm{U}(1)_R$ charge +2 we instead apply the $D^-_\alpha$ derivative to get the same result.) This part of the variation must be cancelled by a 1-$\overline{F}$ term, since we have no other 2-$\overline{F}$ terms. But we cannot get an $\overline{F}$ field from a $D^{\pm}_\alpha$ variation, and thus there is no way to cancel this variation. So there can be no terms with two $F$'s or two $\overline{F}$'s in the action.

**Terms with one $F$:** Now we have to examine terms with one $F$, looking first once again at those with fermions $\psi^{\pm}$, $\overline{\psi}^{\pm}$. The possible terms are

$$(\psi F\psi)\psi^2\,, \qquad (\psi F\psi)\overline{\psi}^2\,, \qquad (\psi F\psi)D^{\pm\pm}\,, \qquad \psi F\,\partial\!\!\!/\overline{\psi}\,, \tag{4.9}$$

[4] We do this because we are using the $D^{\pm}_\alpha$ variation. We could just as well look at the $\overline{D}^{\pm}_\alpha$ variation of the term $D^{\pm\pm}\,tr(F^2)$, since $\overline{D}^{\pm\pm} = D^{\pm\pm}$.



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as well as their conjugates. The $\pm$ indices on the $\psi$'s, which would indicate the total $\mathrm{U}(1)_R$ charge of each term, have not been included. It is not hard to show that there is only one term of each form above for each $\mathrm{U}(1)_R$ charge. The different orderings of the $\pm$ indices does not give new terms, as each ordering can be related to any other using a Fierz identity.

As before, these terms will always have some part of their variation with one more $F$, giving a 2-$F$ term. Since we have shown that there are no 2-$F$ terms in the action, we must cancel this part of the variation with the variation of another 1-$F$ term. In order for two 1-$F$ terms to give the same 2-$F$ variation, they must have the same fields, and so in this case must be the same term. So, the coefficients of all these 1-$F$ action terms with fermions must be zero.

Here we note that this argument does not hold if there is more than one vector multiplet. In that case, there are multiple terms with the same basic field content. For example, look at the terms $\psi_1^- F_2 \psi_1^+$ and $\psi_2^- F_1 \psi_1^+$, where the 1,2 subscripts denote different vector multiplet "flavors". These both give a variation that can be written as $tr(F_1 F_2)\, \psi_{1\alpha}^+$, and thus their supersymmetry variation could possibly cancel. This makes it much more difficult to determine if there are 3-derivative terms with more than one vector multiplet.

Now we move on to single-$F$ terms without fermions, which is a little trickier to explore. The only possible terms are of the form

$$(\not\partial F) \cdot (\not\partial \phi)\,, \qquad (\not\partial F) \cdot (\not\partial \overline{\phi})\,, \qquad (\not\partial \phi) \cdot F \cdot (\not\partial \overline{\phi})\,, \tag{4.10}$$

as well as their conjugates. The two $\not\partial$'s in (4.10) are contracted on dotted indices. The two derivatives must act on different fields, because otherwise they would vanish due to the symmetry of $F$.

Each of these terms requires either $\phi$ or $\overline{\phi}$. First we look at the terms with a $\overline{\phi}$. We write these as

$$(\overline{\sigma}^\mu F \sigma^\nu)\, \overline{\phi}\, f(\phi, \overline{\phi})\, \partial_\mu \partial_\nu\,, \tag{4.11}$$

where the derivatives must act on different fields. Consider now $\overline{D}^+$ acting on the $\overline{\phi}$, or, equivalently, the $D^+$ variation of its conjugate. The conjugate term,

$$(\sigma^\mu \overline{F} \overline{\sigma}^\nu)\, \phi\, f(\overline{\phi}, \phi)\, \partial_\mu \partial_\nu\,, \tag{4.12}$$

includes in its $D^+$ variation the term

$$(\sigma^\mu \overline{F} \overline{\sigma}^\nu)\, \psi_\alpha^+\, f(\overline{\phi}, \phi)\, \partial_\mu \partial_\nu\,. \tag{4.13}$$

We cannot generate an $\overline{F}$ or a $\phi$ from a $D^+$ variation, and the only way to get a $\psi^+$ is from the term we are considering. So, we can only cancel (4.13) by generating a $\overline{\phi}$ from the variation of another term. The variation of $\overline{\psi}^-$ gives $\partial\overline{\phi}$, but this would require a term in the action with an $F$ and fermions, which we have already show not to exist. Thus there can be no term in the action with no fermions, an $\overline{F}$, and a $\phi$, and so also no term with no fermions, an $F$, and a $\overline{\phi}$.

So, the only possible 1-$F$, no fermion term left is

$$(\not\partial F) \cdot (\not\partial \phi) \tag{4.14}$$



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times a function of $\phi$ only, not $\overline{\phi}$. This has a variation that includes

$$(\partial\!\!\!/ F) \cdot (\partial\!\!\!/ \psi^{+}_{\alpha}) \,. \tag{4.15}$$

To get a cancelling variation term we must generate the $F$ field from a $\psi^{-}$,

$$(\partial\!\!\!/ \psi^{+}) \cdot (\partial\!\!\!/ \psi^{-}) \,. \tag{4.16}$$

This has a variation which includes

$$(\partial\!\!\!/ \psi^{+}) \cdot (\partial\!\!\!/ F)_{\alpha} \,. \tag{4.17}$$

While this looks similar to (4.15), in fact, after performing Fierz transformations we find that they differ by a term

$$\partial\!\!\!/_{\alpha\dot{\beta}} \psi^{+\gamma} \, \partial\!\!\!/^{\dot{\beta}\beta} F_{\beta\gamma} \,. \tag{4.18}$$

There is no way to generate such a term from a $D^{+}$ variation, and any Fierz transformation leave us with terms of the form we are trying to cancel in the first place. So, there can be no action terms of this form, and so no action terms with any $F$ fields at all.

**Terms without $F$:** Showing that there are no terms without $F$s but with fermions is very similar to the case with one $F$ and fermions, but even simpler. Acting on a fermion in such a term with the appropriate supersymmetry variation once again gives an $F$, and since we have no 1-$F$ action terms we need to cancel this variation with the variation of another action term without an $F$. So, the $F$ in the variation must be generated in the variation. Since there is only one way to get an $F$ from a given variation, it must come from the same fermion it did before, and so we can only cancel this variation with the same term we generated it from. So, the coefficient of a term with fermions and no $F$'s is zero.

Now all that remains is to show that there are no terms with zero $F$'s and no fermions. The only fields we have left are the $D$'s, the $\phi$'s, and their derivatives. Without $\psi$'s the derivatives must contract with each other, and so we must have two or zero derivatives. This means we must have one or three $D$'s, respectively. The variation of a three-$D$ term cannot be cancelled by a one-$D$ term, and it is easy to see that distinct 3-$D$ terms cannot cancel, so we can have no three-$D$ terms. We can ignore the $\mathrm{U}(1)_R$ charges of the $D$'s now, since only terms with the same net charge can possibly cancel, and so the charges on the $D$'s will just go along for the ride. The only remaining possible terms are then

$$D\partial^2\phi f(\phi,\overline{\phi}) \,, \qquad D\partial^2\overline{\phi} g(\phi,\overline{\phi}) \,, \qquad D\partial\phi\partial\overline{\phi} h(\phi,\overline{\phi}) \,. \tag{4.19}$$

The variation of these terms must then cancel among themselves. A $D^{+}$ variation acting on the scalar functions will give a $\psi$ for each term. These three terms then each have derivatives on different fields, and we cannot move the derivatives around using integration by parts because of the $D$. This means that $f$, $g$, and $h$ must be independent of $\phi$, as well as $\overline{\phi}$, and therefore just constants. Now if we let the variation act on the $D$ field, we get three terms with $\partial\!\!\!/ \psi$ parts, and the other parts will not cancel, and do not form a total derivative. So there are no terms that have neither $F$'s nor $\psi$'s.



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This exhausts all possible terms, and shows that we cannot have any supersymmetric, gauge invariant, three-derivative terms with only one vector multiplet. For reasons discussed above, it seems much more difficult to generalize this argument to theories with more than one vector multiplet. Similarly, this strategy becomes quite cumbersome to use to search for Chern-Simons-like terms at the four-derivative level.

## 5. Global issues on the Coulomb branch

The conclusion of the negative arguments of the last two sections is that the issue of the existence of Chern-Simons-like terms in particular, and of a systematic and effective derivative expansion in general, in $N = 2$ effective actions on the Coulomb branch is problematic. In the $N = 1$ and $N = 0$ cases similar combinatoric problems connected with gauge invariance and the existence of Chern-Simons-like terms arise. But in these cases because the gauge fields can be assigned positive derivative weight, this does not present a problem of principle for the derivative expansion. For $N \geq 2$ theories, however, the gauge potential superfield must be assigned negative weight in the derivative expansion, and the existence of Chern-Simons-like terms becomes a problem of principle for the existence of a systematic derivative expansion.

One positive result of the searches in sections 3 and 4 was the identification of a 4-derivative superspace Chern-Simons-like term globally on the Coulomb branch of $N = 2$ theories, which could not have been found had we worked solely with field strength superfields. (Note that such a term might also survive on the moduli space of $N = 4$ theories, provided that it can be completed to an $N = 4$ supersymmetric multiplet [20].) We will devote the remainder of this section to a discussion of this term and the related issue of global obstructions on the Coulomb branch. This issue has interesting parallels to the recent discussion in [21] of F terms on the moduli space of $N = 1$ theories which are globally obstructed from being written as D-terms.

Recall from section 3 that the term in question is the four-derivative term

$$S_4 = \int d^4x du d^8\theta \, V_a^{--} \, D^+ \left( A_a^b(W, \overline{W}) D^+ W_b \right) + \text{c.c.}\,, \tag{5.1}$$

where $A$ satisfies

$$\partial^c A_a^b - \partial^b A_a^c = 0\,, \tag{5.2}$$

which is the local integrability condition for

$$A_a^b = \partial^b B_a\,, \tag{5.3}$$

for some $B_a(W, \overline{W})$, where $\partial^b \equiv \partial/\partial W_b$ is the holomorphic derivative on the Coulomb branch, $\mathcal{M}$. If (5.3) held, then $S_4$ could be written solely in terms of field strength superfields as $S_4 \sim \int d^4x d^8\theta \, B_a(W, \overline{W}) \, \overline{W}^a$. But (5.2) is only a local integrability condition, so $B_a$ may fail to exist globally on the Coulomb branch.

Indeed, treating $A_a^b$ as the coefficients functions of a set of (1,0)-forms on the Coulomb branch,

$$A_a \equiv A_a^b(W, \overline{W}) \, dW_b\,, \tag{5.4}$$



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condition (5.2) is equivalent to the $A_a$ being closed under the Dolbeault exterior differential $\partial \equiv dW_a \partial^a$,

$$\partial A_a = 0\,, \tag{5.5}$$

which implies, locally, that the $A_a$ are exact,

$$A_a = \partial B_a\,, \tag{5.6}$$

for some (0,0)-forms $B_a$. So, the interesting Chern-Simons-like terms are the non-trivial Dolbeault cohomology classes in $H^{(1,0)}_{\partial}(\mathcal{M})$.

There are, however, some caveats to this description of the global Chern-Simons-like terms, which come from the low energy $\mathrm{U}(1)^n$ gauge invariance. First, since the potential superfield appears explicitly in (5.1), the coefficient functions $A_a^b$, and therefore the one-forms $A_a$, are only defined up to holomorphic linear redefinitions of the $W_a$, instead of general holomorphic changes of variables on $\mathcal{M}$. This is because the gauge invariance of (5.1) under $\delta V_a^{--} = -D^{--}\lambda_a$ does not permit non-linear transformations of the $V_a^{--}$, and the $W_a$ are linearly related to the $V_a^{--}$ by (3.8). We are thus restricted to special coordinates on the Coulomb branch in (5.1).

Secondly, the special coordinates, $W_a$, on $\mathcal{M}$ are not single-valued. They are allowed, by virtue of the electric-magnetic duality ambiguity in the description of $\mathrm{U}(1)^n$ theories, to have monodromies valued in the discrete $\mathrm{Sp}(2n, \mathbb{Z})$ duality group [22]. These monodromies are determined by the 2-derivative terms in the effective action on the Coulomb branch and transform the field strength superfields, $W_a$, nonlinearly and the potential superfields, $V_a^{--}$, in a nonlocal way. This makes the global definition of the Chern-Simons-like term problematic. It would be desirable to have a duality-covariant superspace formalism for the vector multiplets in order to address this issue. For the field strength multiplet, it is not hard to develop such a formalism along the lines of [23]. For the potential superfield, needed in the Chern-Simons-like term (5.1), such a formalism is not known.

However, there are some models in which the electric-magnetic monodromies are trivial and so (5.1) can be used. The simplest example is the Coulomb branch of the scale invariant SU(2) $N = 2$ superQCD with four massless fundamental hypermultiplets. In this case the Coulomb branch is $\mathbb{C}^*$, the complex $W$-plane minus the point at the origin, and the special coordinate $W$ experiences only the $\mathbb{Z}_2$ monodromy $W \to -W$ upon circling the origin (inherited from the un-gauge-fixed center of the SU(2) gauge group). Thus the coefficient function $A$ in (5.1) is constrained to be $\mathbb{Z}_2$-even: $A = A(W^2, \overline{W}^2, W\overline{W})$. Any such $AdW$ is trivially closed under $\partial$; unfortunately, it is also exact. (For example, $AdW = (\overline{W}/W)dW = \partial[\overline{W}\ln(W\overline{W})]$.) Other monodromy-free Coulomb branches occur for scale-invariant theories with $\mathrm{SU}(2)^n$ product gauge groups with fundamental and bi-fundamental matter in various configurations [24]; but all of the resulting non-compact Coulomb branches appear to have trivial $H^{(1,0)}_{\partial}$.

A less familiar set of theories which do have non-trivial cohomology classes are the $N = 2$ theories with compact Coulomb branches discussed in [25, 26, 27]. These theories can be realized as compactifications of 6-dimensional little string theories [28, 29] on $T^2$. In the simplest example [26], the Coulomb branch is given in special coordinates by the $\mathbb{Z}_2$



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orbifold of a complex torus minus the four orbifold fixed points; i.e., the complex $W$-plane with the identifications $W \sim -W$, $W \sim W+1$, and $W \sim W+\Lambda$, for some complex scale $\Lambda$, minus the four points $W \in \{0, \frac{1}{2}, \frac{\Lambda}{2}, \frac{1+\Lambda}{2}\}$. (This is the scale invariant SU(2) model of the last paragraph with toroidally compactified Coulomb branch.) A nontrivial element of $H^{(1,0)}_{\overline{\partial}}$ has a constant coefficient function $A$, so that the (1,0)-form is $\sim dW$ which is not exact since $W$ is not single-valued on $\mathcal{M}$, nor can any function of $\overline{W}$ be added to it to make it single-valued.[5] This thus gives an example of a global Chern-Simons-like term. Inserting this $A$ in (5.1) shows that the existence of this term simply stems from the fact that although $W$ is not single-valued on $T^2$, $D^+W$ is.

The global structure of the Coulomb branch also plays a role in the classification of the holomorphic 4-derivative term listed in (3.11), which has the form

$$\begin{aligned} S_{4a} &= \int d^4x\, d^4\theta\, \partial_\mu W_a \partial^\mu W_b\, \mathcal{G}^{ab}(W) + \text{c.c.}\,, \\ &= \frac{1}{2}\int d^4x\, d^4\theta\, (\overline{D}^+)^2 [D^- W_a \cdot D^- W_b\, \mathcal{G}^{ab}] + \text{c.c.}\,, \end{aligned} \tag{5.7}$$

where in the second line we have used the $N=2$ algebra (3.2) and the chirality of $W$ (3.3). This shows that this term is more properly thought of as an integral over 3/4 of superspace; it is a special case of one of the 3/4 superspace terms found in [4]. Because of its holomorphic nature, terms of this form enjoy a non-renormalization theorem, but only so long as they cannot be rewritten as a nonholomorphic term integrated over the full superspace. The following manipulations give a condition for when this can happen.

$$\begin{aligned} \int d^4x\, d^4\theta\, d^2\overline{\theta}^+ \Big\{ D^- W_a \cdot D^- W_b\, K^{ab} &= -W_a\, (D^-)^2 W_b\, K^{ab} - W_a D^- W_b \cdot D^- W_c\, \partial^c K^{ab} \\ &= -W_a\, (\overline{D}^-)^2 \overline{W}_b\, K^{ab} - D^- W_a \cdot D^- W_b\, W_c \partial^b K^{ac} \Big\}\,, \end{aligned}$$

where in the second line we used the Bianchi identity (3.10). Moving the last term to the left side then gives

$$\int d^4x\, d^4\theta\, d^2\overline{\theta}^+\, D^- W_a \cdot D^- W_b\, \partial^b [W_c K^{ac}] = -8 \int d^4x\, d^8\theta\, W_a \overline{W}_b\, K^{ab}\,. \tag{5.8}$$

Thus, if the $\mathcal{G}$ holomorphic coefficient function in (5.7) satisfies

$$\mathcal{G}^{ab}(W) = \partial^b H^a(W) \tag{5.9}$$

for some holomorphic $H^a = W_c K^{ac}$, then the $S_{4a}$ term can be rewritten as a non-holomorphic 4-derivative term integrated over the whole superspace. With two or more vector multiplets, (5.9) can fail to be integrable even locally, giving examples of locally holomorphic $S_{4a}$ terms.

But, even with just a single vector multiplet, (5.9) can fail to be integrable globally. In this case of a single vector multiplet, the globally holomorphic coefficient function $\mathcal{G}$ in (5.7)

[5] More generally, since $T^2/\mathbb{Z}_2$ minus its fixed points is equivalent to a four-punctured sphere, there is a rich set of analytic functions $g$ on this space, and $A = g(\overline{W})dW$ is non-trivial in cohomology; presumably physical considerations will place limits on the allowed singularities in $g$ at the fixed points.



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can be thought of as defining a section of a holomorphic quadratic form $\mathcal{G}(W)\,(dW)^2$ on the Coulomb branch, $\mathcal{M}$. By (5.9) this global coefficient function is defined up to the equivalence

$$\mathcal{G}(W) \sim \mathcal{G}(W) + \partial H(W)\,, \tag{5.10}$$

which, because of the holomorphy of the functions, is the same as the equivalence in holomorphic de Rham cohomology on the Coulomb branch (as opposed to Dolbeault cohomology). As in our discussion of the Chern-Simons-like term above, when there are non-trivial electric-magnetic duality monodromies on the Coulomb branch the global definition of the $\mathcal{G}$ section is more complicated, and cannot be taken simply as single-valued on $\mathcal{M}$. It is an interesting open question whether the holomorphic 4-derivative terms (5.7) exist by virtue of a global obstruction on one-dimensional Coulomb branches.

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